

Partially 2-Colored Permutations and the Boros-Moll Polynomials

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Abstract

We find a combinatorial setting for the coefficients of the Boros-Moll polynomials $P_m(a)$ in terms of partially 2-colored permutations. Using this model, we give a combinatorial proof of a recurrence relation on the coefficients of $P_m(a)$. This approach enables us to give a combinatorial interpretation of the log-concavity of $P_m(a)$ which was conjectured by Moll and confirmed by Kauers and Paule.

Keywords: partially 2-colored permutation, Boros-Moll polynomial, rising factorial, log-concavity, bijection

AMS Classifications: 05A05; 05A10; 05A20

1 Introduction

The main objective of this paper is to present a combinatorial approach to the log-concavity of the Boros-Moll polynomials. The Boros-Moll polynomials $P_m(a)$ arise in the evaluation of a quartic integral, see [3–7, 13]. Boros and Moll have shown that for any $a > -1$ and any nonnegative integer m ,

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \quad (1.1)$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j} \binom{m-j}{k} \binom{2k+2j}{k+j} \frac{(a+1)^j (a-1)^k}{2^{3(k+j)}}. \quad (1.2)$$

Boros and Moll also derived a single sum formula for $P_m(a)$:

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k, \quad (1.3)$$

which implies that the coefficients of $P_m(a)$ are positive. More precisely, let $d_i(m)$ be the coefficient of a^i in $P_m(a)$. Then (1.3) gives

$$d_i(m) = 2^{-2m} \sum_{k=i}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \quad (1.4)$$

Several proofs of the formula (1.3) can be found in the survey of Amdeberhan and Moll [2].

Further positivity properties of $P_m(a)$ have been studied recently. Boros and Moll [5] have shown that the sequence $\{d_i(m)\}_{0 \leq i \leq m}$ is unimodal for $m \geq 0$. Moll conjectured that this sequence is log-concave, that is, for $m \geq 2$ and $1 \leq i \leq m-1$,

$$d_i^2(m) \geq d_{i-1}(m) d_{i+1}(m). \quad (1.5)$$

This conjecture has been confirmed by Kauers and Paule [12] based on recurrence relations. Chen and Xia [10] have proved a stronger property of $d_i(m)$, called the ratio monotone property, which implies both the log-concavity and the spiral property. Moll [14, 15] posed a conjecture that is stronger than the log-concavity of $P_m(a)$. This conjecture has been proved by Chen and Xia [11]. Chen and Gu [8] established the reverse ultra log-concavity of the Boros-Moll polynomials.

It turns out that the polynomials $P_m(a)$ are closely related to combinatorial structures. The 2-adic valuation of the numbers $i!m!2^{m+i}d_i(m)$ has been studied by Amdeberhan, Manna and Moll [1], and Sun and Moll [16]. By using reluctant functions and an extension of Foata's bijection, Chen, Pang and Qu [9] have found a combinatorial derivation of the single sum formula (1.3) from the double sum formula (1.2). For the special case $a = 1$, we are led to a combinatorial argument for the identity

$$\sum_{k=0}^m 2^{-2k} \binom{2k}{k} \binom{2m-k}{m} = \sum_{k=0}^m 2^{-2k} \binom{2k}{k} \binom{2m+1}{2k}.$$

However, this combinatorial approach does not seem to apply to recurrence relations for $d_i(m)$ or the log-concavity of $P_m(a)$.

In this paper, we shall consider a variation of the coefficients $d_i(m)$, that is,

$$D_i(m) = \binom{2m}{m-i} m!i!(m-i)!2^i d_i(m). \quad (1.6)$$

Then the numbers $D_i(m)$ have a combinatorial interpretation in terms of partially 2-colored permutations.

Using this combinatorial setting, we give an explanation of the following recurrence relation of $d_i(m)$ derived independently by Kauers and Paule [12] and Moll [14]:

$$i(i+1)d_{i+1}(m) = i(2m+1)d_i(m) - (m-i+1)(m+i)d_{i-1}(m). \quad (1.7)$$

The reasoning of the above recurrence relation also implies a simple combinatorial interpretation of the log-concavity of the Boros-Moll polynomials.

2 A combinatorial setting for $D_i(m)$

In this section, we shall give a combinatorial interpretation of $D_i(m)$ by introducing the structure of partially 2-colored permutations. Throughout this paper, we shall adopt the notation $(x)_n$ for rising factorials, that is, $(x)_0 = 1$ and for $n > 0$,

$$(x)_n = x(x+1) \cdots (x+n-1).$$

From the expression (1.4) for $d_i(m)$, we have

$$\begin{aligned} d_i(m) &= 2^{-2m} \sum_{k=i}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i} \\ &= 2^{-2m} \sum_{j=0}^{m-i} 2^{j+i} \binom{2m-2i-2j}{m-i-j} \binom{m+i+j}{i+j} \binom{i+j}{i} \\ &= 2^{-2m} \sum_{j=0}^{m-i} 2^{j+i} \frac{(2m-2i-2j)!}{(m-i-j)!(m-i-j)!} \cdot \frac{(m+i+j)!}{(i+j)!m!} \cdot \frac{(i+j)!}{j!i!} \\ &= 2^{-2m} \sum_{j=0}^{m-i} 2^{j+i} \frac{2^{2m-2i-2j}(m-i-j-\frac{1}{2})!}{(m-i-j)!} \cdot \frac{(m+i+j)!}{(i+j)!m!} \cdot \frac{(i+j)!}{j!i!}. \end{aligned}$$

It follows that

$$\begin{aligned} m!i!(m-i)!2^i d_i(m) &= (m-i)! \sum_{j=0}^{m-i} \left(\frac{1}{2}\right)^j \frac{(m-i-j-\frac{1}{2})!}{(m-i-j)!} \cdot \frac{(m+i+j)!}{j!}, \\ &= \sum_{j=0}^{m-i} \binom{m-i}{j} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)_{m-i-j} (m+i+j)!, \end{aligned}$$

which yields

$$D_i(m) = \binom{2m}{m-i} \sum_{j=0}^{m-i} \binom{m-i}{j} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)_{m-i-j} (m+i+j)!. \quad (2.1)$$

We proceed to give a combinatorial interpretation of $D_i(m)$ according to the expression (2.1). It is well known that $(x)_n$ equals the generating function for permutations on $[n]$ with respect to the number of cycles. Let σ be a permutation on $[n]$. The weight of σ is defined as x^k , where k is the number of cycles in σ . So $(x)_n$ is the weighted count of permutations on $[n]$.

Suppose that (A, B, C) is a composition of $[2m] = \{1, 2, \dots, 2m\}$, namely, any A , B and C are disjoint and $A \cup B \cup C = [2m]$, where A , B and C are allowed to be empty. A permutation on $[2m]$ associated with a composition (A, B, C) of $[2m]$ is called a partially 2-colored permutation on $[2m]$ if it can be written as $(\pi|\sigma)$, where π is a permutation on $A \cup B$ and σ is a permutation on C . We assume that the elements in A are white, the elements in B are black and written in boldface, while the elements in C are uncolored.

Moreover, we need to use two different representations for the permutations π and σ in a partially 2-colored permutation $(\pi|\sigma)$. To be precise, we shall write π in the one-line notation in the form of a sequence. For example, $5, 7, 8, 2, 1, 6, 4, 3$ is the one-line representation of a permutation. On the other hand, we shall express σ in terms of the cycle decomposition. For instance, the permutation in the above example has cycle decomposition $(1, 5)(2, 7, 4)(3, 8)(6)$.

Let $\mathcal{D}_i(m)$ denote the set of all partially 2-colored permutations $(\pi|\sigma)$ on $[2m]$ such that the 2-colored permutation π has $m + i$ black elements. For example, consider the partially 2-colored permutation

$$(\mathbf{2}, \mathbf{12}, 8, \mathbf{11}, \mathbf{5}, \mathbf{9}, \mathbf{7}, 1, \mathbf{4}, \mathbf{3} | (6, 10))$$

in $\mathcal{D}_2(6)$. Then we have $A = \{1, 8\}$, $B = \{2, 3, 4, 5, 7, 9, 11, 12\}$, and $C = \{6, 10\}$. From the definition, we see that for a partially 2-colored permutation $(\pi|\sigma)$ in $\mathcal{D}_i(m)$, we have $|A \cup C| = m - i$.

We are now ready to give a combinatorial interpretation of $D_i(m)$. With respect to the weight a partially 2-colored permutation $(\pi|\sigma)$ in $\mathcal{D}_i(m)$, we impose the following rules:

- (1) An element in A is given a weight $\frac{1}{2}$;
- (2) A cycle in σ is given a weight $\frac{1}{2}$.

The weight $(\pi|\sigma)$ is defined as the product of the weights of the white elements and the cycles. In light of the above weight assignment, $D_i(m)$ can be viewed as a weighted count of partially 2-colored permutations. The weight of a set S means to be the sum of weights of its elements, and is denoted by $w(S)$.

Theorem 2.1. *For $m \geq 1$, $D_i(m)$ equals the weight of $\mathcal{D}_i(m)$.*

Proof. Given a composition (A, B, C) of $[2m]$ such that $|B| = m + i$ and $|A \cup C| = m - i$. Assume that there are j elements in A . It is clear that there are $m - i - j$ elements in C . Now, there are $\binom{2m}{m-i}$ ways to distribute $2m$ elements into B and $A \cup C$. Moreover, there are $\binom{m-i}{j}$ ways to distribute $m - i$ elements into A and C .

Consider partially 2-colored permutations in $\mathcal{D}_i(m)$ associated with composition (A, B, C) of $[2m]$. Since $|A \cup B| = m + i + j$, the sum of weights of permutations on $A \cup B$ equals

$$\left(\frac{1}{2}\right)^j \cdot (m + i + j)!.$$

The weighted sum of permutations on C equals $\left(\frac{1}{2}\right)_{m-i-j}$. This completes the proof. \blacksquare

3 Combinatorial proof of the recurrence relation

Using the interpretation of $D_i(m)$ in terms of partially 2-colored permutation, we give a combinatorial proof for the following recurrence relation of the coefficients $d_i(m)$ of the Boros-Moll polynomials

$$i(i+1)d_{i+1}(m) = i(2m+1)d_i(m) - (m-i+1)(m+i)d_{i-1}(m). \quad (3.1)$$

This recurrence was independently derived by Kauers, Paule [12] and Moll [14].

Utilizing (1.6), the recurrence relation (3.1) can be restated as

$$\frac{1}{2}(m+i+1)D_{i+1}(m) + 2(m-i+1)D_{i-1}(m) = (2m+1)D_i(m). \quad (3.2)$$

To give a combinatorial proof of (3.2), we need to introduce some notation. Let $\mathcal{A}_i(m)$ (resp. $\mathcal{B}_i(m)$ and $\mathcal{C}_i(m)$) denote the set of all partially 2-colored permutations $(\pi|\sigma)$ in $\mathcal{D}_i(m)$ such that exactly one element in A (resp. B and C) is underlined. Obviously, the four sets $\mathcal{A}_i(m)$, $\mathcal{B}_i(m)$, $\mathcal{C}_i(m)$ and $\mathcal{D}_i(m)$ are disjoint. For example,

$$(\mathbf{2}, \mathbf{12}, 8, \mathbf{11}, \mathbf{5}, \underline{\mathbf{9}}, 7, 1, \mathbf{4}, \mathbf{3}|(6, 10))$$

is an underlined partially 2-colored permutation belonging to $\mathcal{B}_2(6)$. By definition and Theorem 2.1, we have

$$(m+i)D_i(m) = w(\mathcal{B}_i(m)), \quad (3.3)$$

$$(m-i)D_i(m) = w(\mathcal{A}_i(m) \cup \mathcal{C}_i(m)). \quad (3.4)$$

Proof. From (3.3) and (3.4), we know that

$$(m+i+1)D_{i+1}(m) = w(\mathcal{B}_{i+1}(m)), \quad (3.5)$$

$$(m - i + 1)D_{i-1}(m) = w(\mathcal{A}_{i-1}(m) \cup \mathcal{C}_{i-1}(m)). \quad (3.6)$$

On the other hand, we have

$$(2m + 1)D_i(m) = w(\mathcal{A}_i(m) \cup \mathcal{B}_i(m) \cup \mathcal{C}_i(m) \cup \mathcal{D}_i(m)). \quad (3.7)$$

First, we claim that

$$\frac{1}{2}w(\mathcal{B}_{i+1}(m)) = w(\mathcal{A}_i(m)). \quad (3.8)$$

Given $(\pi|\sigma) \in \mathcal{B}_{i+1}(m)$ with underlying composition (A, B, C) , where $|B| = m + i + 1$ and $|A \cup C| = m - i - 1$, by changing the underlined black element in π to an underlined white element, we obtain an underlined partially 2-colored permutation in $\mathcal{A}_i(m)$. Clearly, this operation yields a bijection between $\mathcal{B}_{i+1}(m)$ and $\mathcal{A}_i(m)$. Since the weight of a white element equals $1/2$, we obtain (3.8). Substituting i with $i - 1$ in (3.8), we get

$$w(\mathcal{B}_i(m)) = 2w(\mathcal{A}_{i-1}(m)). \quad (3.9)$$

Hence (3.2) simplifies to the following relation

$$2w(\mathcal{C}_{i-1}(m)) = w(\mathcal{C}_i(m) \cup \mathcal{D}_i(m)). \quad (3.10)$$

Assume that $(\pi|\sigma) \in \mathcal{C}_{i-1}(m)$ is a partially 2-colored permutation with underlying composition (A, B, C) , that is, $|B| = m + i - 1$, $|A \cup C| = m - i + 1$, and σ is a permutation with an underlined element. Suppose that σ has cycle decomposition C_0, C_1, \dots, C_r , where C_0 contains the underlined element. Without loss of generality, we may always write C_0 as $(\underline{i_1} i_2 \dots i_k)$. Given $(\pi|\sigma) \in \mathcal{C}_{i-1}(m)$, we define

$$\Delta(\pi|\sigma) = \{\Delta_1, \Delta_2, \dots, \Delta_k\},$$

where

$$\begin{aligned} \Delta_1 &= (\pi, \mathbf{i}_1 | (\underline{i_2}, i_3, \dots, i_k) C_1 C_2 \dots C_r), \\ \Delta_2 &= (\pi, \mathbf{i}_1, i_2 | (\underline{i_3}, \dots, i_k) C_1 C_2 \dots C_r), \\ &\quad \dots \\ \Delta_{k-1} &= (\pi, \mathbf{i}_1, i_2, \dots, i_{k-1} | (\underline{i_k}) C_1 C_2 \dots C_r), \\ \Delta_k &= (\pi, \mathbf{i}_1, i_2, \dots, i_{k-1}, i_k | C_1 C_2 \dots C_r). \end{aligned}$$

For $1 \leq j \leq k - 1$, we have $\Delta_j \in \mathcal{C}_i(m)$ and

$$w(\Delta_j) = \frac{1}{2^{j-1}} w(\pi|\sigma). \quad (3.11)$$

Moreover, we see that $\Delta_k \in \mathcal{D}_i(m)$ and

$$w(\Delta_k) = \frac{1}{2^{k-2}} w(\pi|\sigma). \quad (3.12)$$

Conversely, any partially colored permutation in $\mathcal{C}_i(m) \cup \mathcal{D}_i(m)$ can be obtained from a partially colored permutation in $\mathcal{C}_{i-1}(m)$ by applying the above operation Δ . Thus, we deduce that

$$\Delta(\mathcal{C}_{i-1}(m)) = \mathcal{C}_i(m) \cup \mathcal{D}_i(m), \quad (3.13)$$

where Δ acts on the partially colored permutations in $\mathcal{C}_{i-1}(m)$. Since

$$\sum_{j=1}^{k-1} \frac{1}{2^{j-1}} + \frac{1}{2^{k-2}} = 2,$$

combining (3.11), (3.12) and (3.13) we obtain (3.2). This completes the proof. \blacksquare

4 Combinatorial proof of the log-concavity

In this section, we shall use the structure of partially 2-colored permutations to give a combinatorial reasoning of the following relation

$$(m+i+1)D_{i+1}(m) \cdot (m-i+1)D_{i-1}(m) < (m+i)(m-i+1)D_i^2(m), \quad (4.1)$$

which implies the log-concavity of the Boros-Moll polynomials. We shall follow the notation introduced in the previous section.

Proof. From (3.5) and (3.6), we see that

$$\begin{aligned} & (m+i+1)D_{i+1}(m) \cdot (m-i+1)D_{i-1}(m) \\ &= w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{A}_{i-1}(m) \cup \mathcal{C}_{i-1}(m)) \\ &= w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{A}_{i-1}(m)) + w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{C}_{i-1}(m)). \end{aligned} \quad (4.2)$$

Meanwhile, in view of (3.3) and (3.4), we find

$$\begin{aligned} & (m+i)(m-i+1)D_i^2(m) \\ &= w(\mathcal{B}_i(m)) \cdot w(\mathcal{A}_i(m) \cup \mathcal{C}_i(m) \cup \mathcal{D}_i(m)) \\ &= w(\mathcal{B}_i(m)) \cdot w(\mathcal{A}_i(m)) + w(\mathcal{B}_i(m)) \cdot w(\mathcal{C}_i(m) \cup \mathcal{D}_i(m)). \end{aligned} \quad (4.3)$$

Hence (4.1) can be recast as

$$w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{A}_{i-1}(m)) + w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{C}_{i-1}(m))$$

$$< w(\mathcal{B}_i(m)) \cdot w(\mathcal{A}_i(m)) + w(\mathcal{B}_i(m)) \cdot w(\mathcal{C}_i(m) \cup \mathcal{D}_i(m)). \quad (4.4)$$

Invoking (3.8) and (3.9), we obtain

$$w(\mathcal{B}_{i+1}(m)) \cdot w(\mathcal{A}_{i-1}(m)) = w(\mathcal{B}_i(m)) \cdot w(\mathcal{A}_i(m)). \quad (4.5)$$

Using (4.5) and the fact that

$$2w(\mathcal{C}_{i-1}(m)) = w(\mathcal{C}_i(m) \cup \mathcal{D}_i(m))$$

as given by (3.10), (4.4) simplifies to

$$\frac{1}{2}w(\mathcal{B}_{i+1}(m)) < w(\mathcal{B}_i(m)). \quad (4.6)$$

Applying (3.8), (4.6) is equivalent to the relation

$$w(\mathcal{A}_i(m)) < w(\mathcal{B}_i(m)), \quad (4.7)$$

which can be easily deduced from (3.3) and (3.4), since for $1 \leq i \leq m-1$,

$$w(\mathcal{A}_i(m)) \leq w(\mathcal{A}_i(m) \cup \mathcal{C}_i(m)) = (m-i)D_i(m) < (m+i)D_i(m) = w(\mathcal{B}_i(m)). \quad (4.8)$$

This completes the proof. ■

Acknowledgments. This work was supported by the 973 Project, the National Natural Science Foundation of China, the PCSIRT Project and the Fundamental Research Funds for Central Universities of the Ministry of Education of China.

References

- [1] T. Amdeberhan, D. Manna and V. Moll, The 2-adic valuation of a sequence arising from a rational integral, J. Combin. Theory Ser. A 115 (8) (2008) 1474–1486.
- [2] T. Amdeberhan and V. Moll, A formula for a quartic integral: A survey of old proofs and some new ones, Ramanujan J. 18 (2009) 91–102.
- [3] G. Boros and V. Moll, An integral hidden in Gradshteyn and Ryzhik, J. Comput. Appl. Math. 106 (1999) 361–368.
- [4] G. Boros and V. Moll, A sequence of unimodal polynomials, J. Math. Anal. Appl. 237 (1999) 272–287.
- [5] G. Boros and V. Moll, A criterion for unimodality, Electron. J. Combin. 6 (1999) #R10.

- [6] G. Boros and V. Moll, The double square root, Jacobi polynomials and Ramanujan's Master Theorem, *J. Comput. Appl. Math.* 130 (2001) 337–344.
- [7] G. Boros and V. Moll, *Irresistible Integrals*, Cambridge University Press, New York/Cambridge, 2004.
- [8] W.Y.C. Chen and C.C.Y. Gu, The reverse ultra log-concavity of the Boros-Moll polynomials, *Proc. Amer. Math. Soc.* 137 (2009) 3991–3998.
- [9] W.Y.C. Chen, S.X.M. Pang and E.X.Y. Qu, On the combinatorics of the Boros-Moll polynomials, *Ramanujan J.* 21 (2010) 41–51.
- [10] W.Y.C. Chen and E.X.W. Xia, The ratio monotonicity of the Boros-Moll polynomials, *Math. Comp.* 78 (2009) 2269–2282.
- [11] W.Y.C. Chen and E.X.W. Xia, A proof of Moll's minimum conjecture, *European J. Combin.*, to appear.
- [12] M. Kauers and P. Paule, A computer proof of Moll's log-concavity conjecture, *Proc. Amer. Math. Soc.* 135 (2007) 3847–3856.
- [13] V. Moll, The evaluation of integrals: A personal story, *Notices Amer. Math. Soc.* 49 (3) (2002) 311–317.
- [14] V. Moll, Combinatorial sequences arising from a rational integral, *Online J. Anal. Comb.* 2 (2007) #4 .
- [15] V.H. Moll and D.V. Manna, A remarkable sequence of integers, *Expo. Math.* 27 (2009) 289–312.
- [16] X.Y. Sun and V. Moll, A binary tree representation for the 2-adic valuation of a sequence arising from a rational integral, *Integers* 10 (2009) 211–222.